

UPPER BOUNDS FOR THE BETTI NUMBERS OF A GIVEN HILBERT FUNCTION

Anna Maria Bigatti

Dipartimento di Matematica dell'Università di Genova
Via L.B. Alberti 4, I-16132 Genova Italy
E-mail Bigatti@UniMat.To.CNR.It

Let $R := k[X_1, \dots, X_N]$ be the polynomial ring in N indeterminates over a field k of characteristic 0 with $\deg(X_i) = 1$ for $i = 1, \dots, N$, and let I be a homogeneous ideal of R . The Hilbert function of I is the function from \mathbf{N} to \mathbf{N} which associates to every natural number d the dimension of I_d as a k -vectorspace.

I has an essentially unique minimal graded free resolution

$$0 \longrightarrow L_m \xrightarrow{d_m} L_{m-1} \xrightarrow{d_{m-1}} \dots \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0 \xrightarrow{d_0} I \longrightarrow 0$$

which is characterized, among the free graded resolutions, by the condition

$$d_q(L_q) \subseteq (X_1, \dots, X_N)L_{q-1} \quad \forall q \geq 1$$

And therefore the Betti numbers, which are defined by

$$\beta_q(I) := \text{rank} L_q$$

are invariants of I .

From Macaulay [M] (see also Robbiano [R]) it follows that a lex-segment ideal has the greatest number of generators (the 0-th Betti number β_0) among all the homogeneous ideals with the same Hilbert function.

In this paper we prove that this fact extends to every Betti number, in the sense that all the Betti numbers of a lex segment ideal are bigger than or equal to the ones of any homogeneous ideal with the same Hilbert function.

Section 1 gives some useful notation and definitions and many simple properties of Borel normed sets.

In Section 2 a Theorem is derived (Theorem 2.1) which is our main tool in comparing lex-segment and Borel normed sets.

In Section 3, using a result due to Eliahou and Kervaire [E-K], we compare lex-segment and Borel normed ideals, and then, using some results due to Galligo [Ga] and Möller-Mora [M-M], we compare lex-segment and homogeneous ideals.

Section 4 gives the formula which computes the Betti numbers of the lex-segment ideal, given its Hilbert function, and these are the sharp upper bounds for the Betti numbers of any homogeneous ideal with the same Hilbert function.

1. Some remarks on Borel normed sets.

Notations. Let \mathbf{X}_N denote the set of indeterminates $\{X_1, \dots, X_N\}$; then $(\mathbf{X}_N)^D$ indicates the set of all monomials of degree D in \mathbf{X}_N .

Let \mathbf{S} be a subset of $(\mathbf{X}_N)^D$; then $\mathbf{X}_N\mathbf{S}$ denotes the multiples of \mathbf{S} of degree $D + 1$, i.e. $\mathbf{X}_N\mathbf{S} = \cup_{T \in \mathbf{S}} \{X_1T, \dots, X_NT\}$.

If $T = X_1^{t_1} \dots X_N^{t_N}$, then we denote by $m(T) := \max\{i \mid t_i > 0\}$, i.e. the largest index of the indeterminates actually occurring in T .

Definition. A set of monomials $\mathbf{S} \subseteq (\mathbf{X}_N)^D$ is **Borel normed** if $T \in \mathbf{S}$ implies $X_i \frac{T}{X_j} \in \mathbf{S}$ for all j such that X_j divides T and for all $i < j$.

Definition. On $(\mathbf{X}_N)^D$ we will use the **lexicographic order**, i.e. if $T = X_1^{t_1} \dots X_N^{t_N}$ and $T' = X_1^{s_1} \dots X_N^{s_N}$ are two monomials in $(\mathbf{X}_N)^D$ we will say that $T > T'$ if $t_1 = s_1, \dots, t_{i-1} = s_{i-1}$ and $t_i > s_i$.

Note that it is a total ordering and then there exists the minimum of every subset of $(\mathbf{X}_N)^D$.

Lemma 1.1. Let \mathbf{S} be a Borel normed set.

Then $X_i(\min \mathbf{S}) \in \mathbf{X}_N(\mathbf{S} \setminus \{\min \mathbf{S}\}) \iff i < m(\min \mathbf{S})$.

Proof. Let $T := \min \mathbf{S}$.

‘ \implies ’ : If $X_i T \in \mathbf{X}_N(\mathbf{S} \setminus \{T\})$ then $X_i T = X_j T'$ where $T' \in \mathbf{S} \setminus \{T\}$

Thus $T' > T$, hence $i < j$.

On the other hand X_j divides T and so $j \leq m(T)$.

Then $i < m(T)$.

‘ \impliedby ’ : If $i < m(T)$ then, since \mathbf{S} is Borel normed, $T' := X_i \frac{T}{X_{m(T)}} \in \mathbf{S} \setminus \{T\}$. Hence $X_i T = X_{m(T)} T' \in \mathbf{X}_N(\mathbf{S} \setminus \{T\})$. ■

Proposition 1.2. Let \mathbf{S} be a Borel normed set.

Then $\mathbf{X}_N \mathbf{S} = \cup_{T \in \mathbf{S}} \{X_{m(T)} T, \dots, X_N T\}$ and this is a disjoint union (i.e. $\{X_{m(T)} T, \dots, X_N T\} \cap \{X_{m(T')} T', \dots, X_N T'\} = \emptyset \quad \forall T' \neq T$).

Proof. By induction on the cardinality of the set:

$|\mathbf{S}| = 1$: Since \mathbf{S} is Borel normed it follows that $\mathbf{S} = \{X_1^D\}$. Then $\mathbf{X}_N \mathbf{S} = \{X_1 X_1^D, \dots, X_N X_1^D\}$.

$|\mathbf{S}| > 1$: Let the thesis be true if the cardinality is smaller than $|\mathbf{S}|$.

Let $T' := \min \mathbf{S}$. Then $\mathbf{X}_N \mathbf{S} = \{X_1 T', \dots, X_N T'\} \cup \mathbf{X}_N(\mathbf{S} \setminus \{T'\})$.

But from Lemma 1.1 we have $X_i T' \in \mathbf{X}_N(\mathbf{S} \setminus \{T'\}) \quad \forall i < m(T')$.

Hence $\mathbf{X}_N \mathbf{S} = \{X_{m(T')} T', \dots, X_N T'\} \cup \mathbf{X}_N(\mathbf{S} \setminus \{T'\})$.

Note that $\mathbf{S} \setminus \{T'\}$ is a Borel normed set and then, by the inductive hypothesis: $\mathbf{X}_N(\mathbf{S} \setminus \{T'\}) = \cup_{T \in \mathbf{S} \setminus \{T'\}} \{X_{m(T)} T, \dots, X_N T\}$.

Therefore $\mathbf{X}_N \mathbf{S} = \cup_{T \in \mathbf{S}} \{X_{m(T)}T, \dots, X_NT\}$.

Moreover from Lemma 1.1 we have that if $i \geq m(T')$ then $X_iT' \notin \mathbf{X}_N(\mathbf{S} \setminus \{T'\})$ and so $\{X_{m(T')}T', \dots, X_NT'\} \cap \mathbf{X}_N(\mathbf{S} \setminus \{T'\}) = \emptyset$.

So, in particular, $\{X_{m(T')}T', \dots, X_NT'\} \cap \{X_{m(T)}T, \dots, X_NT\} = \emptyset$
 $\forall T > T'$. ■

Definition. Define

$$m_i(\mathbf{S}) := |\{T \in \mathbf{S} \mid m(T) = i\}|$$

i.e. the number of the elements of \mathbf{S} which “finish” with X_i , and similary

$$m_{\leq i}(\mathbf{S}) := |\{T \in \mathbf{S} \mid m(T) \leq i\}|$$

i.e. the number of the elements of \mathbf{S} in the first i indeterminates.

Proposition 1.3. Let \mathbf{S} be a Borel normed set, then

- i) $m_i(\mathbf{X}_N \mathbf{S}) = m_{\leq i}(\mathbf{S})$.
- ii) $|\mathbf{X}_N \mathbf{S}| = \sum_{i=1}^N m_{\leq i}(\mathbf{S})$.

Proof.

- i) From Proposition 1.2 we have that $\cup_{T \in \mathbf{S}} \{X_{m(T)}T, \dots, X_NT\}$ is a disjoint union. Then, in such a representation of $\mathbf{X}_N \mathbf{S}$ every monomial T with $m(T) = i$ can be uniquely expressed as X_iT' where $T' \in \mathbf{S}$ and $i \geq m(T')$. Therefore there exists a 1-1 correspondence between the monomials T in $\mathbf{X}_N \mathbf{S}$ with $m(T) = i$ and the monomials T' in \mathbf{S} with $m(T') \leq i$. Hence $m_i(\mathbf{X}_N \mathbf{S}) = m_{\leq i}(\mathbf{S})$.

- ii) $|\mathbf{X}_N \mathbf{S}| = \sum_{i=1}^N m_i(\mathbf{X}_N \mathbf{S}) = \sum_{i=1}^N m_{\leq i}(\mathbf{S})$. ■

Definition. A set of monomials $\mathbf{S} \subseteq (\mathbf{X}_N)^D$ is **lex-segment** if

$$T \in \mathbf{S} \text{ and } T' > T \implies T' \in \mathbf{S}$$

or equivalently \mathbf{S} is a lex-segment set if and only if $\mathbf{S} = \{T \mid T \geq \min \mathbf{S}\}$.

Remark. \mathbf{S} is a lex-segment set $\implies \mathbf{S}$ is a Borel normed set.

Definition. We can uniquely decompose \mathbf{S} , with respect to X_N , as follows

$$\mathbf{S} = \mathbf{S}_0 \cup X_N \mathbf{S}_1 \cup X_N^2 \mathbf{S}_2 \cup \dots \cup X_N^D \mathbf{S}_D$$

where the \mathbf{S}_d 's are sets of monomials in $N - 1$ indeterminates.

More precisely

$$\mathbf{S}_d \subseteq (\mathbf{X}_{N-1})^{D-d}$$

Remark. It is easy to see that $\forall i < N \quad m_{\leq i}(\mathbf{S}) = m_{\leq i}(\mathbf{S}_0)$.

In particular, $m_{\leq N-1}(\mathbf{S}) = |\mathbf{S}_0|$.

Proposition 1.4.

- i) Let \mathbf{S} be a Borel normed (lex-segment) set in N indeterminates. For every d \mathbf{S}_d is a Borel normed (lex-segment) set in $N - 1$ indeterminates.
- ii) If \mathbf{S} is a Borel normed (lex-segment) set in $(\mathbf{X}_N)^D$ then $\mathbf{X}_N \mathbf{S}$ is a Borel normed (lex-segment) set in $(\mathbf{X}_N)^{D+1}$.

Proof. Easy exercise.

Lemma 1.5. \mathbf{S} is a Borel normed set $\iff \mathbf{S}_d$ is a Borel normed set $\forall d$,
and $\mathbf{X}_{N-1} \mathbf{S}_d \subseteq \mathbf{S}_{d-1} \quad \forall d > 0$.

Proof.

‘ \implies ’: \mathbf{S}_d is Borel normed set follows from Proposition 1.4.i.

Then it remains to prove that $\mathbf{X}_{N-1} \mathbf{S}_d \subseteq \mathbf{S}_{d-1} \quad \forall d > 0$.

Let $T \in \mathbf{S}_d$ i.e. $X_N^d T \in \mathbf{S}$. Since \mathbf{S} is Borel normed we have:

$$X_N^{d-1} X_i T = X_i \frac{X_N^d T}{X_N} \in \mathbf{S} \quad \forall i < N.$$

Hence $X_i T \in \mathbf{S}_{d-1} \quad \forall i < N$. Thus $\mathbf{X}_{N-1} \mathbf{S}_d \subseteq \mathbf{S}_{d-1}$.

‘ \impliedby ’: We need to prove: $T \in \mathbf{S} \implies X_i \frac{T}{X_j} \in \mathbf{S} \quad \forall i < j$ and $X_j | T$:

Let $T = X_N^d T'$ where $T' \in \mathbf{S}_d$. Then:

Since \mathbf{S}_d is a Borel normed set it follows that

$X_i \frac{T}{X_j} = X_N^d \left(X_i \frac{T'}{X_j} \right) \in \mathbf{S} \quad \forall i < j < N, X_j | T.$
 Since $\mathbf{X}_{N-1} \mathbf{S}_d \subseteq \mathbf{S}_{d-1} \quad \forall d > 0$ (and then $X_N | T$) it follows that
 $X_i \frac{T}{X_N} = X_N^{d-1} (X_i T') \in \mathbf{S} \quad \forall i < N.$ ■

Definition. Given a set $\mathbf{S} \subseteq (\mathbf{X}_N)^D$ we can uniquely define a corresponding **lex-segment with respect to X_N** (denoted \mathbf{S}^*) as follows:

Recall that $\mathbf{S}_d \subseteq (\mathbf{X}_{N-1})^{D-d}$ then denote by \mathbf{S}_d^* the lex-segment set in $(\mathbf{X}_{N-1})^{D-d}$ with $|\mathbf{S}_d^*| = |\mathbf{S}_d|$, i.e. the set of the greatest $|\mathbf{S}_d|$ monomials in $(\mathbf{X}_{N-1})^{D-d}$.

Then $\mathbf{S}^* := X_N^0 \mathbf{S}_0^* \cup \dots \cup X_N^D \mathbf{S}_D^*.$

Remark. $m_{\leq N-1}(\mathbf{S}) = |\mathbf{S}_0| = |\mathbf{S}_0^*| = m_{\leq N-1}(\mathbf{S}^*).$

Lemma 1.6. Let \mathbf{S} be a Borel normed set. If $m_{\leq i}(\mathbf{S}_d^*) \leq m_{\leq i}(\mathbf{S}_d) \quad \forall i \leq N-1$ and $\forall d$, then \mathbf{S}^* is a Borel normed set.

Proof. By Lemma 1.5 it suffices to show that \mathbf{S}_d^* is a Borel normed set $\forall d$ and $\mathbf{X}_{N-1} \mathbf{S}_d^* \subseteq \mathbf{S}_{d-1}^* \quad \forall d > 0.$

The fact that \mathbf{S}_d^* is a Borel normed set is obvious since \mathbf{S}_d^* is a lex-segment set.

It remains to prove $\mathbf{X}_{N-1} \mathbf{S}_d^* \subseteq \mathbf{S}_{d-1}^*:$

From Proposition 1.3.ii it follows that for every Borel normed set \mathbf{S} $|\mathbf{X}_N \mathbf{S}| = \sum_{i=1}^N m_{\leq i}(\mathbf{S}).$ But, by hypothesis we have
 $|\mathbf{X}_{N-1} \mathbf{S}_d^*| = \sum_{i=1}^{N-1} m_{\leq i}(\mathbf{S}_d^*) \leq \sum_{i=1}^{N-1} m_{\leq i}(\mathbf{S}_d) = |\mathbf{X}_{N-1} \mathbf{S}_d| \quad \forall d.$
 Since \mathbf{S} is Borel normed, it follows from Lemma 1.5 $\mathbf{X}_{N-1} \mathbf{S}_d \subseteq \mathbf{S}_{d-1} \quad \forall d > 0.$ Hence

$$|\mathbf{X}_{N-1} \mathbf{S}_d^*| \leq |\mathbf{X}_{N-1} \mathbf{S}_d| \leq |\mathbf{S}_{d-1}| = |\mathbf{S}_{d-1}^*| \quad \forall d$$

Thus, since $\mathbf{X}_{N-1} \mathbf{S}_d^*$ and \mathbf{S}_{d-1}^* are lex-segments sets (Proposition 1.4), we have

$$\mathbf{X}_{N-1} \mathbf{S}_d^* \subseteq \mathbf{S}_{d-1}^* \quad \forall d$$

■

Remark. We will see (in Theorem 2.1) that the hypothesis of this Lemma are always verified. Thus for every Borel normed set \mathbf{S} we have that \mathbf{S}^* is Borel normed.

Definition. Let $T = (X_1^{t_1}, \dots, X_N^{t_N}) \in (\mathbf{X}_N)^D$. Define the **corresponding monomial** \overline{T} in $(\mathbf{X}_{N-1})^D$ as follows:

$$\overline{T} := (X_1^{\overline{t_1}}, \dots, X_{N-1}^{\overline{t_{N-1}}})$$

$$\text{where } \overline{t_i} := t_i \quad \forall i < N-1 \quad \text{and} \quad \overline{t_{N-1}} := t_{N-1} + t_N$$

or equivalently

$$\overline{T} := T \left(\frac{X_{N-1}}{X_N} \right)^{t_N}$$

Lemma 1.7.

- i) If $T, T' \in (\mathbf{X}_N)^D$ $T \leq T'$ then $\overline{T} \leq \overline{T'}$.
- ii) Let \mathbf{S} be a Borel normed set then $\overline{\min \mathbf{S}} = \min \mathbf{S}_0$.

Proof.

- i) Let $T = (X_1^{t_1}, \dots, X_N^{t_N})$, $T' = (X_1^{s_1}, \dots, X_N^{s_N})$ and $T < T'$, then we have $t_i = s_i \quad \forall i < j$ and $t_j < s_j$.
Note that $j \neq N$ since otherwise $D = \deg T = \sum t_i < \sum s_i = \deg T' = D$.

Then: $\overline{t_i} = t_i = s_i = \overline{s_i} \quad \forall i < j$ and, (two cases)

if $j = N-1$: $\overline{t_{N-1}} = \deg T - \sum_{i=1}^{N-2} t_i = \overline{s_{N-1}}$ then $\overline{T} = \overline{T'}$;

if $j < N-1$: $\overline{t_j} = t_j < s_j = \overline{s_j}$ then $\overline{T} < \overline{T'}$.

- ii) Obviously $\min \mathbf{S} \leq \min \mathbf{S}_0$.

Hence from i) it follows that $\overline{\min \mathbf{S}} \leq \overline{\min \mathbf{S}_0} = \min \mathbf{S}_0$.

On the other hand, since \mathbf{S} is Borel normed, $\overline{\min \mathbf{S}} \in \mathbf{S}_0$.

Thus $\overline{\min \mathbf{S}} = \min \mathbf{S}_0$. ■

2. Comparisons between lex-segment and Borel normed sets.

Theorem 2.1. Let \mathbf{L} be a lex-segment set and \mathbf{B} a Borel normed set in $(\mathbf{X}_N)^D$ such that $|\mathbf{L}| \leq |\mathbf{B}|$. Then

$$m_{\leq i}(\mathbf{L}) \leq m_{\leq i}(\mathbf{B}) \quad \forall i = 1, \dots, N$$

Proof. By induction on the number of indeterminates:

$$N = 2: m_{\leq 2}(\mathbf{L}) = |\mathbf{L}| \leq |\mathbf{B}| = m_{\leq 2}(\mathbf{B}) \quad \text{and} \quad m_{\leq 1}(\mathbf{L}) = 1 = m_{\leq 1}(\mathbf{B}).$$

$N > 2$: Inductive hypothesis: let the thesis be true in $N - 1$ indeterminates, and then study for all $i = 1 \dots N$ the relations between $m_{\leq i}(\mathbf{L})$ and $m_{\leq i}(\mathbf{B})$.

$$i = N: m_{\leq N}(\mathbf{L}) = |\mathbf{L}| \leq |\mathbf{B}| = m_{\leq N}(\mathbf{B}).$$

$i = N - 1$: We need to prove:

$$m_{\leq N-1}(\mathbf{L}) \leq m_{\leq N-1}(\mathbf{B})$$

i.e.

$$|\mathbf{L}_0| \leq |\mathbf{B}_0|$$

From the definition of the lex-segment with respect to X_N we have $|\mathbf{B}_0| = |\mathbf{B}^*_0|$. So it will be enough to prove

$$|\mathbf{L}_0| \leq |\mathbf{B}^*_0|.$$

Now \mathbf{B}^*_d and \mathbf{B}_d are, $\forall d$, respectively lex-segment and Borel normed sets of monomials in $N - 1$ indeterminates with the same cardinality. Then by the inductive hypothesis it follows

$$m_{\leq i}(\mathbf{B}^*_d) \leq m_{\leq i}(\mathbf{B}_d) \quad \forall i = 1, \dots, N - 1 \quad \forall d$$

Then, since \mathbf{B} is Borel normed, it follows from Lemma 1.6 that \mathbf{B}^* is Borel normed.

Now recall the definition of corresponding monomial in $(\mathbf{X}_{N-1})^D$ and consider $\overline{\min \mathbf{B}^*}$ and $\overline{\min \mathbf{L}}$.

Note that $\min \mathbf{L} \geq \min \mathbf{B}^*$ (otherwise, since \mathbf{L} is lex-segment, $\mathbf{B}^* \subset \mathbf{L}$ and then $|\mathbf{B}| = |\mathbf{B}^*| < |\mathbf{L}|$), therefore from Lemma 1.7.i

$$\overline{\min \mathbf{L}} \geq \overline{\min \mathbf{B}^*}$$

It follows from Lemma 1.7.ii that, since \mathbf{L} and \mathbf{B}^* are Borel normed

$$\overline{\min \mathbf{L}} = \min \mathbf{L}_0 \quad , \quad \overline{\min \mathbf{B}^*} = \min \mathbf{B}^*_0$$

and then

$$\min \mathbf{L}_0 \geq \min \mathbf{B}^*_0$$

Moreover, since \mathbf{B}^* is lex-segment w.r.to X_N , we have that \mathbf{B}^*_0 is a lex-segment set in $(\mathbf{X}_{N-1})^D$. From these facts it follows that

$$\mathbf{L}_0 \subseteq \mathbf{B}^*_0$$

Hence

$$|\mathbf{L}_0| \leq |\mathbf{B}^*_0|$$

$i < N - 1$: From the case $i = N - 1$ we have $|\mathbf{L}_0| \leq |\mathbf{B}_0|$ where \mathbf{L}_0 and \mathbf{B}_0 are respectively lex-segment and Borel normed sets in $N - 1$ indeterminates. By the inductive hypothesis

$$m_{\leq i}(\mathbf{L}_0) \leq m_{\leq i}(\mathbf{B}_0) \quad \forall i = 1, \dots, N - 1$$

So

$$m_{\leq i}(\mathbf{L}) = m_{\leq i}(\mathbf{L}_0) \leq m_{\leq i}(\mathbf{B}_0) = m_{\leq i}(\mathbf{B}) \quad \forall i < N - 1$$

■

Corollary 2.2. $|\mathbf{L}| = |\mathbf{B}| \implies |\mathbf{X}_N \mathbf{L}| \leq |\mathbf{X}_N \mathbf{B}|$.

Proof. $|\mathbf{X}_N \mathbf{L}| = \sum_{i=1}^N m_{\leq i}(\mathbf{L}) \leq \sum_{i=1}^N m_{\leq i}(\mathbf{B}) = |\mathbf{X}_N \mathbf{B}|$.

■

Definition. Let \mathbf{S} be any set of monomials, then define

$$b_q(\mathbf{S}) := \sum_{T \in \mathbf{S}} \binom{m(T) - 1}{q}$$

Proposition 2.3. Let $\mathbf{B} \subseteq (\mathbf{X}_N)^D$ be a Borel normed set, then

$$b_q(\mathbf{B}) = \binom{N-1}{q} |\mathbf{B}| - \sum_{i=1}^{N-1} \left[m_{\leq i}(\mathbf{B}) \binom{i-1}{q-1} \right]$$

Proof.

$$\begin{aligned}
\sum_{T \in \mathbf{B}} \binom{m(T) - 1}{q} &= \sum_{i=1}^N \left[m_i(\mathbf{B}) \binom{i-1}{q} \right] = \\
&= \sum_{i=1}^N \left[(m_{\leq i}(\mathbf{B}) - m_{\leq i-1}(\mathbf{B})) \binom{i-1}{q} \right] = \\
&= \sum_{i=1}^N \left[m_{\leq i}(\mathbf{B}) \binom{i-1}{q} \right] - \sum_{i=0}^{N-1} \left[m_{\leq i}(\mathbf{B}) \binom{i}{q} \right] = \\
&= \binom{N-1}{q} m_{\leq N}(\mathbf{B}) + \sum_{i=1}^{N-1} \left[m_{\leq i}(\mathbf{B}) \left(\binom{i-1}{q} - \binom{i}{q} \right) \right] = \\
&= \binom{N-1}{q} |\mathbf{B}| - \sum_{i=1}^{N-1} \left[m_{\leq i}(\mathbf{B}) \binom{i-1}{q-1} \right]
\end{aligned}$$

■

Corollary 2.4. Let \mathbf{L} be a lex-segment set and \mathbf{B} a Borel normed set in $(\mathbf{X}_N)^D$ such that $|\mathbf{L}| = |\mathbf{B}|$, then:

- i) $b_q(\mathbf{L}) \geq b_q(\mathbf{B})$;
- ii) $b_q(\mathbf{X}_N \mathbf{L}) \leq b_q(\mathbf{X}_N \mathbf{B})$.

Proof. From Theorem 2.1 we have $m_{\leq i}(\mathbf{L}) \leq m_{\leq i}(\mathbf{B}) \quad \forall i = 1, \dots, N$.

Then:

i)

$$\begin{aligned}
b_q(\mathbf{L}) &= \binom{N-1}{q} |\mathbf{L}| - \sum_{i=1}^{N-1} \left[m_{\leq i}(\mathbf{L}) \binom{i-1}{q-1} \right] \geq \\
&\geq \binom{N-1}{q} |\mathbf{B}| - \sum_{i=1}^{N-1} \left[m_{\leq i}(\mathbf{B}) \binom{i-1}{q-1} \right] = b_q(\mathbf{B})
\end{aligned}$$

- ii) Recall from Proposition 1.3.i that if \mathbf{S} is a Borel normed set then $m_i(\mathbf{X}_N \mathbf{S}) = m_{\leq i}(\mathbf{S})$. Thus:

$$b_q(\mathbf{X}_N \mathbf{L}) = \sum_{T \in \mathbf{X}_N \mathbf{L}} \binom{m(T) - 1}{q} = \sum_{i=1}^N m_i(\mathbf{X}_N \mathbf{L}) \binom{i-1}{q} =$$

$$= \sum_{i=1}^N m_{\leq i}(\mathbf{L}) \binom{i-1}{q} \leq \sum_{i=1}^N m_{\leq i}(\mathbf{B}) \binom{i-1}{q} = b_q(\mathbf{X}_N \mathbf{B})$$

■

3. Comparisons between lex-segment and homogeneous ideals.

Definition. Let I be a monomial ideal in $k[X_1, \dots, X_N]$. Then we denote by $\mathbf{G}(I)$ the minimal system of generators of I , i.e. the set of all monomials in I which are not proper multiples of any monomial in I , and by $\mathbf{G}_k(I_d)$ the basis of I_d as a k -vectorial space.

Definition. A monomial ideal I in $k[X_1, \dots, X_N]$ is called:

- i) **Lex-segment** if $\mathbf{G}_k(I_d)$ is a lex-segment set $\forall d$;
- ii) **Borel normed** if $T \in I \implies X_i \frac{T}{X_j} \in I \quad \forall i < j$ such that $X_j | T$, or equivalently if $\mathbf{G}_k(I_d)$ is a Borel normed set $\forall d$;
- iii) **Stable** if $T \in I \implies X_i \frac{T}{X_{M(T)}} \in I$.

Remark. If I is a monomial ideal. Then

$$I \text{ is lex-segment} \implies I \text{ is Borel normed} \implies I \text{ is stable.}$$

Theorem 3.1. Eliahou-Kervaire(1987). Let I be a stable ideal, then

$$\beta_q(I) = \sum_{T \in \mathbf{G}(I)} \binom{m(T)-1}{q}$$

Corollary 3.2. Let I be a stable ideal.

$$\text{Then } \beta_q(I) = \sum_{d \geq 0} [b_q(\mathbf{G}_k(I_d)) - b_q(\mathbf{X}_N \mathbf{G}_k(I_{d-1}))].$$

Proof. From Theorem 3.1 we have $\beta_q(I) = b_q(\mathbf{G}(I)) = \sum_{d \geq 0} b_q((\mathbf{G}(I))_d)$.

Then, since $(\mathbf{G}(I))_d = \mathbf{G}_k(I_d) \setminus \{X_N \mathbf{G}_k(I_{d-1})\}$, the thesis follows. ■

Corollary 3.3. Let $I^{\mathbf{L}}$ be a lex-segment ideal and $I^{\mathbf{B}}$ a Borel normed ideal with the same Hilbert function, then

$$\beta_q(I^{\mathbf{L}}) \geq \beta_q(I^{\mathbf{B}})$$

Proof. Note that for all d $\mathbf{G}_k(I_d^{\mathbf{L}})$ and $\mathbf{X}_N \mathbf{G}_k(I_d^{\mathbf{L}})$ are lex-segment sets, and $\mathbf{G}_k(I_d^{\mathbf{B}})$ and $\mathbf{X}_N \mathbf{G}_k(I_d^{\mathbf{B}})$ are Borel normed sets.

Moreover $I^{\mathbf{L}}$ and $I^{\mathbf{B}}$ have the same Hilbert function, i.e. $\forall d$

$$|\mathbf{G}_k(I_d^{\mathbf{L}})| = H_{I^{\mathbf{L}}}(d) = H_{I^{\mathbf{B}}}(d) = |\mathbf{G}_k(I_d^{\mathbf{B}})|$$

From Corollary 2.4, we then have

$$b_q(\mathbf{G}_k(I_d^{\mathbf{L}})) \geq b_q(\mathbf{G}_k(I_d^{\mathbf{B}})) \text{ and } b_q(\mathbf{X}_N \mathbf{G}_k(I_d^{\mathbf{L}})) \leq b_q(\mathbf{X}_N \mathbf{G}_k(I_d^{\mathbf{B}})) \quad \forall d$$

and, from Corollary 3.2

$$\begin{aligned} \beta_q(\mathbf{G}(I^{\mathbf{L}})) &= \sum_{d>0} [b_q(\mathbf{G}_k(I_d^{\mathbf{L}})) - b_q(\mathbf{G}_k(I_{d-1}^{\mathbf{L}}))] \geq \\ &\geq \sum_{d>0} [b_q(\mathbf{G}_k(I_d^{\mathbf{B}})) - b_q(\mathbf{G}_k(I_{d-1}^{\mathbf{B}}))] = \beta_q(\mathbf{G}(I^{\mathbf{B}})) \end{aligned}$$

■

Remark. Note that for every Borel normed ideal I there exists a lex segment ideal with the same Hilbert function as that of I . In fact let \mathbf{S}_d be the lex-segment set in $(\mathbf{X}_N)^d$ with $|\mathbf{S}_d| = H_I(d)$. Then, from Corollary 2.2,

$$|\mathbf{X}_N \mathbf{S}_d| \leq |\mathbf{X}_N \mathbf{G}_k(I^{\mathbf{B}})|$$

Since $I^{\mathbf{B}}$ is an ideal we have $\mathbf{X}_N \mathbf{G}_k(I^{\mathbf{B}})_d \subseteq \mathbf{G}_k(I^{\mathbf{B}})_{d+1}$. Thus

$$|\mathbf{X}_N \mathbf{S}_d| \leq |\mathbf{X}_N \mathbf{G}_k(I^{\mathbf{B}})_d| \leq H_I(d+1) = |\mathbf{S}_{d+1}|$$

Since $\mathbf{X}_N \mathbf{S}_d$ and \mathbf{S}_{d+1} are lex segments we get $\mathbf{X}_N \mathbf{S}_d \subseteq \mathbf{S}_{d+1}$.

Hence we can consider the \mathbf{S}_d 's as the basis of the part in degree d of an ideal that is lex-segment and has the same Hilbert function of I .

Theorem 3.4. Galligo(1974).

Let I be a homogeneous ideal in $k[X_1, \dots, X_N]$ and let σ a term-ordering. There exists a Zariski open subset $U \subseteq GL(N)$ such that for every $g \in U$, $Lt_{\sigma}(g(I))$ is invariant under the action of the Borel subgroup $B(N)$ of $GL(N)$. In particular, if $\text{char}(k) = 0$, then $Lt_{\sigma}(g(I))$ is Borel normed.

Remark. In this way we can obtain for every homogeneous ideal I , an ideal $I^{\mathbf{B}}$ with the same Hilbert function and the same Betti numbers as those of I , and such that $Lt_{\sigma}(I^{\mathbf{B}})$ is Borel normed.

Theorem 3.5. Macaulay(1927).

Let I be a homogeneous ideal in $k[X_1, \dots, X_N]$ and let σ a term-ordering. Then

$$H_I = H_{Lt_{\sigma}(I)}$$

Remark. Let I be a homogeneous ideal in $k[X_1, \dots, X_N]$. Then there exists a lex segment ideal with the same Hilbert function as that of I .

In fact let $I^{\mathbf{B}}$ be the ideal obtained from I by a generic change of coordinates (Theorem 3.4). We have that $Lt_{\sigma}(I^{\mathbf{B}})$ is a Borel normed ideal and hence there exists a lex segment ideal with Hilbert function $H_{Lt_{\sigma}(I^{\mathbf{B}})} = H_{I^{\mathbf{B}}}$ (Theorem 3.5).

Theorem 3.6. Möller-Mora(1983).

Let I be a homogeneous ideal in $k[X_1, \dots, X_N]$ and let σ a term-ordering. Then

$$\beta_I \leq \beta_{Lt_{\sigma}(I)}$$

Theorem 3.7. Let I be a homogeneous ideal and let $I^{\mathbf{L}}$ be the lex-segment ideal with the same Hilbert function as that of I . Then for all q

$$\beta_q(I^{\mathbf{L}}) \geq \beta_q(I)$$

Proof. Let $I^{\mathbf{B}}$ be the ideal obtained by Theorem 3.4. Then

$$H_{I^{\mathbf{L}}} = H_I = H_{I^{\mathbf{B}}} \quad \text{and} \quad \beta_q(I^{\mathbf{B}}) = \beta_q(I)$$

From Macaulay's Theorem it follows that

$$H_{I^{\mathbf{B}}} = H_{Lt_{\sigma}(I^{\mathbf{B}})}$$

Then, from Corollary 3.3

$$\beta_q(I^{\mathbf{L}}) \geq \beta_q(Lt_{\sigma}(I^{\mathbf{B}}))$$

From Möller-Mora's Theorem

$$\beta_q(Lt_\sigma(I^{\mathbf{B}})) \geq \beta_q(I^{\mathbf{B}})$$

and then

$$\beta_q(I^{\mathbf{L}}) \geq \beta_q(I)$$

■

4. Upper Bounds for Betti Numbers.

Theorem 4.1. Let I be a Borel normed ideal and, with abuse of notation, let I_d denote $\mathbf{G}_k(I_d)$. Then

$$\begin{aligned} \beta_q(I) &= \\ &= \binom{N-1}{q} |I_D| - \sum_{i=1}^{N-1} m_{\leq i}(I_D) \binom{i-1}{q-1} - \sum_{d=1}^{D-1} \left[\sum_{i=1}^{N-1} \left[m_{\leq i}(I_d) \binom{i}{q} \right] \right] \end{aligned}$$

Where D is the largest degree of a generator of I .

Proof. From Corollary 3.2 it follows that

$$\begin{aligned} \beta_q(I) &= \sum_{d=1}^D [b_q(I_d) - b_q(\mathbf{X}_N I_{d-1})] = \\ &= b_q(I_D) + \sum_{d=1}^{D-1} [b_q(I_d)] - \sum_{d=0}^{D-1} [b_q(\mathbf{X}_N I_d)] = \\ &= b_q(I_D) + \sum_{d=1}^{D-1} [b_q(I_d) - b_q(\mathbf{X}_N I_d)] = \\ &= b_q(I_D) + \sum_{d=1}^{D-1} \left[\sum_{i=1}^N \left[m_i(I_d) \binom{i-1}{q} \right] - \sum_{i=1}^N \left[m_i(\mathbf{X}_N I_d) \binom{i-1}{q} \right] \right] \end{aligned}$$

Since I_d is a Borel normed set it follows from Proposition 1.3.i that $m_i(\mathbf{X}_N I_d) = m_{\leq i}(I_d)$, $\forall d$. Then $\beta_q(I) =$

$$= b_q(I_D) + \sum_{d=1}^{D-1} \left[\sum_{i=1}^N \left[(m_i(I_d) - m_{\leq i}(I_d)) \binom{i-1}{q} \right] \right] =$$

$$\begin{aligned}
&= b_q(I_D) - \sum_{d=1}^{D-1} \left[\sum_{i=1}^N \left[m_{\leq i-1}(I_d) \binom{i-1}{q} \right] \right] = \\
&= b_q(I_D) - \sum_{d=1}^{D-1} \left[\sum_{i=1}^{N-1} \left[m_{\leq i}(I_d) \binom{i}{q} \right] \right] = \\
&= \binom{N-1}{q} |I_D| - \sum_{i=1}^{N-1} m_{\leq i}(I_D) \binom{i-1}{q-1} - \sum_{d=1}^{D-1} \left[\sum_{i=1}^{N-1} \left[m_{\leq i}(I_d) \binom{i}{q} \right] \right]
\end{aligned}$$

■

Definition. It is well known (see Robbiano [R]) that, if h and n are positive integers, then h can be written uniquely in the form

$$h = \binom{h(n)}{n} + \binom{h(n-1)}{n-1} + \dots + \binom{h(i)}{i}$$

where $h(n) > h(n-1) > \dots > h(i) \geq i \geq 1$.

This unique expression is called **binomial expansion** of h in base n and it is denoted by h_n , and define

$$(h_n)_t^s := \binom{h(n)+s}{n+t} + \binom{h(n-1)+s}{n-1+t} + \dots + \binom{h(i)+s}{i+t}$$

The particular significance of the binomial expansion of the values of the Hilbert function becomes apparent when we attend to write an explicit formula which computes the Betti numbers of a lex-segment ideal:

Let \mathbf{S} be a lex-segment set in $(\mathbf{X}_N)^D$ and let d be the largest integer such that $X_1^{D-d} X_N^d \in \mathbf{S}$.

Since \mathbf{S} is a lex-segment set, \mathbf{S} contains all the monomials

$$X_1^{D-d} \{X_1, \dots, X_N\}^d$$

The number of these elements is $\binom{N+d-1}{N-1}$ which is exactly the first binomial in the binomial expansion of $H(D)$ in base $N-1$.

The set of the remaining monomials of \mathbf{S} is strictly contained in

$$X_1^{D-d-1} \{X_2, \dots, X_N\}^{d+1}$$

Thus, we can think of it as a lex-segment set (strictly contained) in $\{X_2, \dots, X_N\}^{d+1}$. So, repeating the reasoning, we obtain the whole binomial expansion. ■

Proposition 4.2.(Macaulay). Let I be a lex-segment ideal. Then

$$|\mathbf{X}_N \mathbf{G}_k(I_D)| = (H_I(D)_{N-1})^1$$

Proof. As we saw before, the first binomial of the binomial expansion of $H(D)$ in base $N - 1$, $\binom{N+d-1}{N-1}$, represents the number of monomials in $\{X_1, \dots, X_N\}^D$. Thus the multiples of these elements are a set with $\binom{N+(d+1)-1}{N-1}$ elements.

And so on. ■

Proposition 4.3. Let I be a lex-segment ideal. Then

$$m_{\leq i}(I_d) = \binom{H(d)_{N-1}}{-(N-i)}^{-(N-i)}$$

(Where $\binom{h}{n} := 0$ if $n < 0$).

Proof. As before, $\binom{N+d-1}{N-1}$ is the number of monomials in $\{X_1, \dots, X_N\}^D$. Among these, the elements which use only the first i indeterminates number $\binom{i+d-1}{i-1}$ i.e.

$$\binom{N+d-1-(N-i)}{N-1-(N-i)}$$

And so on. ■

Remark. If I is a homogeneous ideal we can calculate the largest degree of a generator of the lex-segment ideal with the same Hilbert function. In fact, Green [Gr] proved that $D + 1$ is the smallest integer greater than the maximum degree of a generator of I for which $H_I(D)^1 = H_I(D + 1)$.

Hence Theorem 4.1 and Proposition 4.3 give a formula which computes the Betti numbers of a lex-segment ideal.

They are sharp upper bounds for homogeneous ideals with the same Hilbert function.

In particular, to count the first syzigies, it is possible to give a simpler formula.

Corollary 4.4. $\beta_1(I) =$

$$(N-1)H(D) - (H(D)_{N-1})_{-1} + \sum_{d=1}^{D-1} \left[(N-1)(H(d)_{N-1})_{-1} - (H(d)_{N-1})_{-2} \right]$$

Proof. $b_1(\mathbf{S}) = \sum_{i=1}^N m_i(\mathbf{S})(i-1) = \sum_{i=1}^N (m_{\leq i}(\mathbf{S}) - m_{\leq i-1}(\mathbf{S}))(i-1) =$
 $\sum_{i=1}^N \left[m_{\leq i}(\mathbf{S})(i-1) \right] - \sum_{i=1}^{N-1} \left[m_{\leq i}(\mathbf{S})i \right] = (N-1)|\mathbf{S}| - \sum_{i=1}^{N-1} m_{\leq i}(\mathbf{S}).$

From Proposition 4.3 it is easy to see that $\sum_{i=1}^{N-1} m_{\leq i}(\mathbf{S}) = (|\mathbf{S}|_{N-1})_{-1}$.
Then $b_1(\mathbf{S}) = (N-1)|\mathbf{S}| - (|\mathbf{S}|_{N-1})_{-1}$.

From Proposition 4.2 it follows that $|\mathbf{X}_N \mathbf{S}| = (|\mathbf{S}|_{N-1})^1$ and from Corollary 3.2 that $\beta_1(I) = \sum_{d>0} [b_1(\mathbf{G}_k(I_d)) - b_1(\mathbf{X}_N \mathbf{G}_k(I_{d-1}))]$.

Hence

$$\begin{aligned} \beta_1 &= \\ &= \sum_{d=1}^D [(N-1)|I_d| - (|I_d|_{N-1})_{-1} - ((N-1)|\mathbf{X}_N I_{d-1}| - (|\mathbf{X}_N I_{d-1}|_{N-1})_{-1})] = \\ &= \sum_{d=1}^D [(N-1)H(d) - (H(d)_{N-1})_{-1} - \\ &\quad ((N-1)(H(d-1)_{N-1})^1 - (H(d-1)_{N-1})_{-1}^1)] \end{aligned}$$

The thesis follows easily. ■

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